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On the torus automorphisms: analytic solution, computability and quantization ☆

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Abstract

The exact solution of the classical torus automorphism, which partial case is Arnold Cat map is obtained and compared with the numerical solution. The torus, considered as the classical phase space admits the quantization in terms of the Weyl pair. The remarkable fact is that quantum map, as the evolution with respect to the discrete time, preserves the Weyl commutation relation. We have obtained also the operator solution of this quantum torus automorphism. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

We consider automorphisms of the 2-torus $Y = [0,1) \times [0,1)$ given by the formula

$$S : Y \to Y : \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad (\mathrm{mod}\, 1), \quad n \in Z, \tag{1}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2}$$

The matrix A belongs to the group $SL(2,Z)$, i.e., a, b, c, d, are integers and

$$\det A = 1. \tag{3}$$

These area preserving maps result usually as Poincaré sections or stroboscopic observations of Hamiltonian systems and they are well-known prototypes for the study of integrability and chaos.

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \tag{4}$$

defines the Cat map [1–3] which is a well-known prototype for chaotic behavior due to hyperbolic structure. In fact it is a Bernoulli system in the bottom of the ergodic hierarchy [1,4] with positive Kolmogorov–Sinai entropy equal to its Lyapunov exponent 0.962.

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In this paper we explicitly construct the analytic solution of the torus automorphisms, Section 2. For the chaotic Cat map we compare in Section 3 the exact solution with the periodic and the DO LOOP approximate solutions. Based on the obtained analytic solution we discuss in Section 4 the Weyl quantization of the torus automorphism and we give the explicit solution of the Heisenberg equations of motion.

2. The explicit solution of the torus automorphism

In order to obtain the explicit solution of (1) it is convenient to work with the coordinates on the torus

$$u_n = e^{2\pi i x_n}, \quad v_n = e^{2\pi i y_n}, \quad |u_n| = |v_n| = 1. \tag{5}$$

The map equation (1) in these variables takes the form:

$$\begin{aligned} u_{n+1} &= u_n^a v_n^b, \\ v_{n+1} &= u_n^c v_n^d. \end{aligned} \tag{6}$$

Using Eqs. (6) we have

$$u_{n+2} = u_{n+1}^a v_{n+1}^b = u_{n+1}^a \left(u_n^c v_n^d\right)^b = u_{n+1}^a u_n^{bc} (v_n)^d. \tag{7}$$

From the first equation of (6) we can express v_n^b via the product $u_{n+1}u_n^{-a}$:

$$u_{n+2} = u_{n+1}^a u_n^{bc} \left(u_{n+1} u_n^{-a}\right)^d = u_{n+1}^{a+d} u_n^{bc-ad}. \tag{8}$$

The exponent of u_n in (8) is just $-\det A = -1$ and the exponent of u_{n+1} is the trace $r = a + d$ of A.

Therefore we have

$$u_{n+2} u_n = u_{n+1}^r. \tag{9}$$

Hence it is enough to solve Eq. (9) and using (6) to obtain the solution for v_n.

It is straightforward to see that the second variable v_n obeys Eq. (9) as well.

The following two lemmas are needed to obtain the explicit solution.

Lemma 1. *The product of two solutions of the recursion relation* (9) *is also a solution of* (9).

Proof. The proof is straightforward. □

Lemma 2. *Let p is a root of the quadratic equation*

$$p^2 - rp + 1 = 0. \tag{10}$$

Then the expression

$$u_n = \xi^{p^n}, \quad \xi \in C, \tag{11}$$

is a solution of (9).

Proof.

$$u_{n+2}u_n u_{n+1}^{-r} = \xi^{p^{n+2}+p^n-rp^{n+1}} = \xi^{p^n(p^2-rp+1)} = 1. \tag{12}$$

As Eq. (10) has two roots $p,\ p^{-1}$, the general solution of (9) is

$$u_n = \xi^{p^n} \eta^{p^{-n}}. \tag{13}$$

In order to satisfy the condition $|u_n| = 1$, which follows from (3) we have to take $|\xi| = |\eta| = 1$. Taking into account this normalization, we can write the solution of (6) in the following form:

$$\begin{aligned} u_n &= e^{i\alpha p^n + i\beta p^{-n}}, \\ v_n &= e^{i\gamma p^n + i\delta p^{-n}}, \end{aligned} \tag{14}$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ should be fixed by the initial conditions. The final form of the solution, parameterized by the variables u and v is:

$$u_n = u_0^{-z_{n-1}}\left(u_0^a v_0^b\right)^{z_n},\\ v_n = v_0^{-z_{n-1}}\left(u_0^c v_0^d\right)^{z_n}, \tag{15}$$

where we have introduced the notation

$$z_n = \frac{p^n - p^{-n}}{p - p^{-1}}. \tag{16}$$

Using (10) we may express z_n in terms of r:

$$z_n = \sum_{m=0}^{(n-1)/2} \frac{n!}{(2m+1)!(n-2m-1)!}\,\frac{1}{2^{n-1}}\,r^{n-2m-1}(r^2-4)^m. \tag{17}$$

This is straightforward

$$p^n = \frac{1}{2^n}\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} r^{n-k}(r^2-4)^{k/2}, \tag{18}$$

$$p^{-n} = \frac{1}{2^n}\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} r^{n-k}(r^2-4)^{k/2}(-1)^k. \tag{19}$$

By subtracting (19) from (18) we notice that for even k the terms are eliminated and only the odd k are left, so after the substitution $k = 2m+1$ we have

$$p^n - p^{-n} = \frac{1}{2^{n-1}}\sum_{m=0}^{(n-1)/2} \frac{n!}{(n-2m-1)!(2m+1)!} r^{n-2m-1}(r^2-4)^m(r^2-4)^{1/2}. \tag{20}$$

We also have

$$p - p^{-1} = \frac{1}{2}\left[r + (r-4)^{1/2}\right] - \frac{1}{2}\left[r - (r-4)^{1/2}\right] = (r-4)^{1/2}. \tag{21}$$

Thus

$$z_n = \frac{p^n - p^{-n}}{p - p^{-1}} = \sum_{m=0}^{(n-1)/2} \frac{n!}{(2m+1)!(n-2m-1)!}\,\frac{1}{2^{n-1}}\,r^{n-2m-1}(r^2-4)^m. \tag{22}$$

Taking the logarithm of both sides of the (13) we obtain the solution of the map in the terms of $x_n,\ y_n$:

$$x_n = -x_0 z_{n-1} + (ax_0 + by_0)z_n, \quad \operatorname{mod} 1,\\ y_n = -y_0 z_{n-1} + (cx_0 + dy_0)z_n, \quad \operatorname{mod} 1. \tag{23}$$

In the case $r^2 \leqslant 4$ we can represent r and p as

$$r = p + p^{-1} = 2\cos\phi, \quad p = e^{i\phi}. \tag{24}$$

The trace r should be an integer, therefore only five values of ϕ are permitted:

$$\phi = 0, \frac{2\pi}{6}, \frac{2\pi}{3}, \frac{\pi}{2}, \pi. \tag{25}$$

The corresponding p are roots of unity. It is sufficient to consider only the primitive roots:

$$p = 1,\ \ p = e^{2\pi i/6},\ \ p = e^{2\pi i/3},\ \ p = i,\ \ p = -1. \tag{26}$$

The solution of the map in these special cases is also given by the general form (23). For $p = e^{i\phi}$ we have

$$\frac{p^n - p^{-n}}{p - p^{-1}} = \frac{\sin(n\phi)}{\sin\phi}. \tag{27}$$

The case $p = 1$ corresponds to $\phi \to 0$, therefore using the Hopital rule we have

$$\frac{p^n - p^{-n}}{p - p^{-1}} = n. \tag{28}$$

The case $p = -1$ corresponds to $\phi \to \pi$:

$$\frac{p^n - p^{-n}}{p - p^{-1}} = (-1)^n n. \tag{29}$$

The other values bring no difficulties and the solution in form (17) applies directly.

For Cat map (4) the trace is:

$$r = \mathrm{tr} A = 3 \tag{30}$$

with trace $r = 3$. Therefore $r^2 > 4$ and

$$p = \frac{3 + \sqrt{5}}{2}. \tag{31}$$

The solution is

$$u_n = u_0^{-z_{n-1}} \left(u_0^a v_0^b\right)^{z_n} = u_0^{-z_{n-1}} (u_0 v_0)_n^z, \tag{32}$$

$$v_n = v_0^{-z_{n-1}} \left(u_0^c v_0^d\right)^{z_n} = v_0^{-z_{n-1}} \left(u_0 v_0^2\right)_n^z, \tag{33}$$

or

$$x_n = -x_0 z_{n-1} + (x_0 + y_0) z_n, \tag{34}$$

$$y_n = -y_0 z_{n-1} + (x_0 + 2y_0) z_n, \tag{35}$$

where z_n is given by (16)

$$z_n = \sum_{m=0}^{(n-1)/2} \frac{n!}{(2m+1)!(n-2m-1)!} \frac{1}{2^{n-1}} 3^{n-2m-1} 5^m. \qquad \square \tag{36}$$

3. Numerical results

Let us consider a digital image with dimension $N \times N$. We shall show that applying any torus automorphism (1) on the image yields a periodic approximation of the dynamical system.

Indeed, let us divide the unit square into N parts. Then the coordinates of any point on the lattice are $x = k/N,\ y = l/N$, with $k,\ l$ integers. Applying the torus automorphism formula (1) on this point yields:

$$x_1 = \alpha \frac{k}{N} + \beta \frac{l}{N}, \quad \mathrm{mod}\, 1, \tag{37}$$

$$y_1 = c \frac{k}{N} + d \frac{l}{N}, \quad \mathrm{mod}\, 1. \tag{38}$$

Since $k,\ l,\ \alpha,\ \beta,\ c,\ d$ are integers, $x_1,\ y_1$ also belong to the lattice. We therefore have $x_1 = k_1/N,\ y_1 = l_1/N$ with

$$k_1 = \alpha k + \beta l, \quad \mathrm{mod}\, N, \tag{39}$$

$$l_1 = ck + dl, \quad \mathrm{mod}\, N. \tag{40}$$

As any automorphism of a finite set is periodic, the restriction of any torus automorphism (1) on the $N \times N$ lattice is a periodic map providing an exact approximation of the original torus automorphism.

The period T of the periodic map can be obtained from the equations:

$$k_n = -k z_{n-1} + (\alpha k + \beta l) z_n, \quad \mathrm{mod}\, N, \tag{41}$$

$$l_n = -l z_{n-1} + (ck + dl) z_n, \quad \mathrm{mod}\, N, \tag{42}$$

with $k, l = 0, 1, \ldots, N$.

If T is the period, then for every (k, l) we have

$$k_T = k, \tag{43}$$

$$l_T = l, \tag{44}$$

which yields

$$k(z_{n-1} + 1) = (\alpha k + \beta l)z_n, \quad \mathrm{mod}\, N, \tag{45}$$

$$l(z_{n-1} + 1) = (ck + dl)z_n, \quad \mathrm{mod}\, N. \tag{46}$$

The solution of system (45), (46) is a highly nontrivial problem in number theory. However, the period T as function of the size N of the $N \times N$ lattice can be computed numerically. For the Cat map it is presented in Fig. 1. Fig. 2 shows the successive transformations of a digital image ($N = 72$) with period 12. Dyson and Falk [5] calculated the upper and lower bounds for the period as functions of the dimension of the image (Fig. 3) and they also related the period to the Fibonacci numbers.

As we obtained the analytic solution of the Cat map, we can compare it to a simple DO LOOP calculation and to the periodic approximation (Fig. 4). The results are identical up to the 18th iteration, after which we see that a significant difference appears among the three different methods. The exact analytic calculation goes very fast beyond the capacity of any conventional computer (25 iterations, in our case Pentium 166 MHz, 32 MB). The predictability horizon with respect to our computer for both the DO LOOP and the periodic approximation is $\tau \simeq 18$, while for the analytic calculation is about 25 iterations and afterwards the process stops.

4. The Weyl quantization of the torus automorphism

So far we have obtained the general solution of the torus automorphism, considering the variables u_n, v_n as classical. In order to quantize the torus automorphism we consider the operators U_n and V_n which satisfy the Weyl commutation relation:

$$U_n V_n = q V_n U_n, \tag{47}$$

where q is some complex number. We postulate that the quantum dynamical evolution is governed by the noncommutative version of (6):

$$\begin{aligned} U_{n+1} &= U_n^a V_n^b, \\ V_{n+1} &= U_n^c V_n^d. \end{aligned} \tag{48}$$

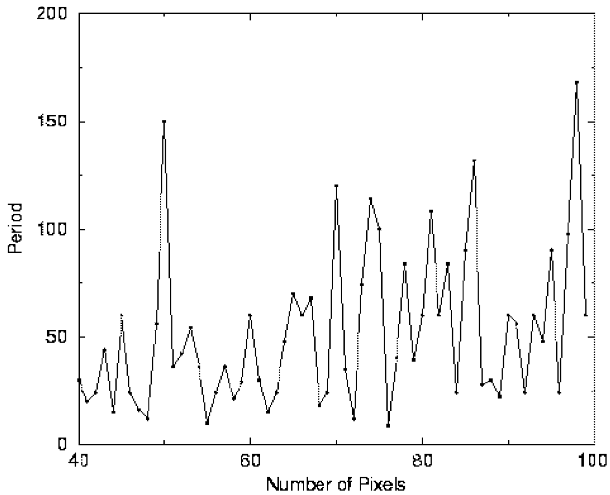

Fig. 1. The period T of the lattice approximation of the Cat map as a function of the size N of the lattice. The number of pixels is $N \times N$.

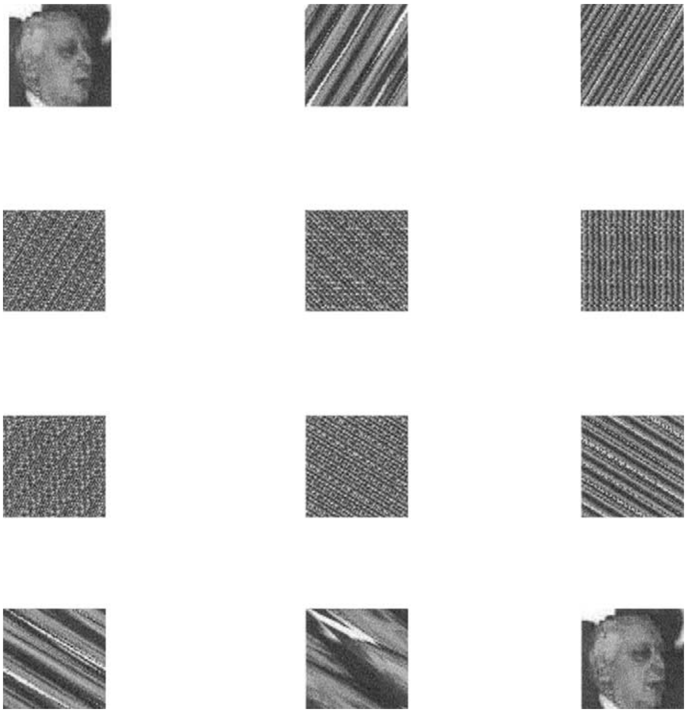

Fig. 2. Successive transformation of a digital image of Ilya Prigogine ($N = 72$) with period $T = 12$.

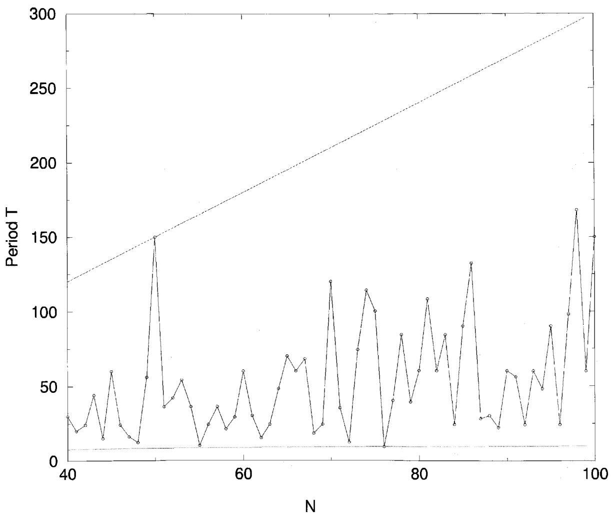

Fig. 3. The dashed lines represent the upper and lower bounds estimated by Dyson and Falk [5] as functions of the size of the image.

It is remarkable that evolution (48) for the operators U_n and V_n preserves the commutation relation (47). Indeed, from (48) we easily obtain

$$U_n^k V_n^l = q^{kl} V_n^l U_n^k. \tag{49}$$

Therefore

$$\begin{aligned} U_{n+1}V_{n+1} &= U_n^a V_n^b U_n^c V_n^d = q^{-bc} U_n^{a+c} V_n^{b+d}, \\ V_{n+1}U_{n+1} &= U_n^c V_n^d U_n^a V_n^b = q^{-ad} U_n^{a+c} V_n^{b+d}, \end{aligned} \tag{50}$$

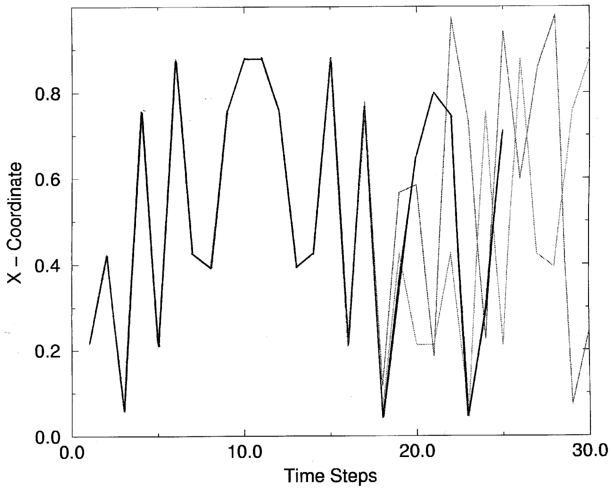

Fig. 4. Comparison of different computation procedures, namely exact analytic computation (solid), periodic approximation (dots), simple DO LOOP computation (dashed).

or

$$U_{n+1}V_{n+1} = q^{ad-bc}V_{n+1}U_{n+1} = qV_{n+1}U_{n+1}. \tag{51}$$

Due to this remarkable property, map (48) is the quantum discrete dynamics $(U_n, V_n) \mapsto (U_{n+1}, V_{n+1})$ of the operators $U_n,\ V_n$ in the Heisenberg picture. This quantum system was considered in [6]. So, the quantum operators $U_n,\ V_n$ and their evolution are indeed the quantization of the classical torus automorphism. With this interpretation, the phase space of the classical system is compact (torus). Therefore its quantization should lead to a finite-dimensional Hilbert space [7]. As a result the constant q should be an Nth root of unity, $q^N = 1$. Only in this case the Weyl commutation relation (47) admits an N-dimensional representation.

As is well known, in the quantum case there arises an ambiguity in the ordering of the operators in the r.h.s. of (48), but this does not affect the conservation of the Weyl commutation relation while the evolution. In the sequel we shall consider only the order of the operators U_n and V_n, shown in (48).

5. Explicit solution of the quantum Cat map

We shall look for the solution of the quantum Cat map (48) in terms of the initial data $U,\ V$. Let us consider the parameterization:

$$\begin{aligned} U_n &= A_n U^{\alpha_n} V^{\beta_n}, \\ V_n &= B_n U^{\gamma_n} V^{\delta_n}, \end{aligned} \tag{52}$$

where $A_n,\ B_n,\ \alpha_n,\ \beta_n,\ \gamma_n,\ \delta_n$ are complex numbers, and substitute (52) into (48). As a result we obtain the following equations:

$$\begin{aligned} A_{n+1} &= A_n^a B_n^b q^{-\alpha_n\beta_n\frac{(a-1)(a-2)}{2}-\gamma_n\delta_n\frac{(b-1)(b-2)}{2}-ab\beta_n\gamma_n}, \\ B_{n+1} &= A_n^c B_n^d q^{-\alpha_n\beta_n\frac{(c-1)(c-2)}{2}-\gamma_n\delta_n\frac{(d-1)(d-2)}{2}-cd\beta_n\gamma_n}, \end{aligned} \tag{53}$$

$$\begin{aligned} \alpha_{n+1} &= a\alpha_n + b\gamma_n, \\ \gamma_{n+1} &= c\alpha_n + d\gamma_n, \end{aligned} \tag{54}$$

$$\begin{aligned} \beta_{n+1} &= a\beta_n + b\delta_n, \\ \delta_{n+1} &= c\beta_n + d\delta_n. \end{aligned} \tag{55}$$

The initial data for this set of variables is the following:

$$\begin{aligned} &A_0 = B_0 = 1,\\ &\alpha_0 = \delta_0 = 1,\\ &\beta_0 = \gamma_0 = 0. \end{aligned} \tag{56}$$

As Eqs. (54) and (55) coincide with Eq. (1) without any limitations on the range of the variables, we can use the solution of the classical map. Taking the logarithm of (15) and substituting the initial conditions (56) we obtain:

$$\begin{aligned} &\alpha_n = az_n - z_{n-1},\\ &\gamma_n = cz_n,\\ &\beta_n = bz_n,\\ &\delta_n = dz_n - z_{n-1}. \end{aligned} \tag{57}$$

Due to the symplectic nature of the equations, the following bilinear form is conserved:

$$\alpha_n\delta_n - \beta_n\gamma_n = \alpha_0\delta_0 - \beta_0\gamma_0 = 1. \tag{58}$$

To verify (58) we need the following identity for the z_n:

$$z_n^2 + z_{n-1}^2 - rz_nz_{n-1} = 1. \tag{59}$$

From (16) z_n satisfies the equation

$$z_{n+1} + z_{n-1} = rz_n. \tag{60}$$

Now, having the solution for the $\alpha,\ \beta,\ \gamma,\ \delta$ we can substitute it in Eq. (53) for A_n and B_n and rewrite it in the following form:

$$\begin{aligned} &A_{n+1} = A_n^aB_n^bq^{-R_n},\\ &B_{n+1} = A_n^cB_n^dq^{-T_n}, \end{aligned} \tag{61}$$

where T_n and R_n are given by:

$$\begin{aligned} &R_n = tz_n^2 + rz_nz_{n-1},\\ &T_n = uz_n^2 + vz_nz_{n-1}, \end{aligned} \tag{62}$$

and

$$\begin{aligned} &e = \frac{1}{2}ac(a-1)(a-2) + \frac{1}{2}bd(b-1)(b-2) + ab^2c,\\ &f = -\frac{1}{2}[c(a-1)(a-2) + b(b-1)(b-2)],\\ &g = \frac{1}{2}ac(c-1)(c-2) + \frac{1}{2}bd(d-1)(d-2) + bc^2d,\\ &h = -\frac{1}{2}[c(c-1)(c-2) + b(d-1)(d-2)]. \end{aligned} \tag{63}$$

Apparently the solution of (61) is an exponential of q, and therefore we shall look for it in the following form:

$$\begin{aligned} &A_n = q^{l_n},\\ &B_n = q^{m_n}. \end{aligned} \tag{64}$$

Substitution of (64) into (61) gives the system of equations

$$\begin{aligned} &l_{n+1} = al_n + bm_n - \left(ez_n^2 + fz_nz_{n-1}\right),\\ &m_{n+1} = cl_n + dm_n - \left(gz_n^2 + hz_nz_{n-1}\right), \end{aligned} \tag{65}$$

which in contradistinction with (1) is an inhomogeneous map. The solution of the linear inhomogeneous equation is the sum of some solution of the inhomogeneous plus the general solution of the homogeneous equations. The homogeneous solution is already obtained for the classical map, so the problem is to find some solution of the inhomogeneous system. Let us search for it in the following form:

$$\begin{aligned} l_n &= \alpha z_n^2 + \beta z_n z_{n-1} + \mu, \\ m_n &= \gamma z_n^2 + \delta z_n z_{n-1} + \nu. \end{aligned} \tag{66}$$

Now, we can substitute parameterization (61) into (60) and use Eqs. (59) and (60) to express z_{n+1} via z_n and z_{n-1}. After this we obtain the linear system of equations for the unknown constants $\alpha,\ \beta,\ \gamma,\ \delta,\ \mu,\ \nu$:

$$\begin{aligned} &\alpha(a+1-r^2) - \beta r + \gamma b = e, \\ &\alpha r + \beta(a+1) + \delta b = f, \\ &\alpha c + \gamma(d+1-r^2) - \delta r = g, \\ &\beta c + \gamma r + \delta(d+1) = h, \\ &\alpha + \mu(1-a) - \nu b = 0, \\ &\gamma - \mu c + \nu(1-d) = 0. \end{aligned} \tag{67}$$

This system is nondegenerate for $r \neq 2, -1$ which are special cases and should be considered separately, as in the classical case. The last two equations in (67) may be solved, giving

$$\begin{aligned} \mu &= \frac{1}{r-2}[-\alpha(d-1) + \gamma b], \\ \nu &= \frac{1}{r-2}[\alpha c - \gamma(a-1)]. \end{aligned} \tag{68}$$

Having the solution of the first four equations (67), we can write the expression for l_n and m_n in the following form:

$$\begin{aligned} l_n &= \alpha z_n^2 + \beta z_n z_{n-1} + \frac{1}{r-2}[-\alpha(d-1) + \gamma b] - E z_{n-1} + (aE + bF) z_n, \\ m_n &= \gamma z_n^2 + \delta z_n z_{n-1} + \frac{1}{r-2}[\alpha c - \gamma(a-1)] - F z_{n-1} + (cE + dF) z_n, \end{aligned} \tag{69}$$

where we have added the solution of homogeneous system (66), parameterized by the constants E and F. Imposing the initial data (56), we define these constants:

$$\begin{aligned} E &= \frac{1}{r-2}[\alpha(d-1) - \gamma b], \\ F &= \frac{1}{r-2}[\gamma(a-1) - \alpha c], \end{aligned} \tag{70}$$

so finally

$$l_n = \alpha z_n^2 + \beta z_n z_{n-1} + \frac{1}{r-2}[-\alpha(d-1) + \gamma b](z_{n-1} + 1) + \frac{1}{r-2}[\alpha(1-a) - \gamma b] z_n, \tag{71}$$

$$m_n = \gamma z_n^2 + \delta z_n z_{n-1} + \frac{1}{r-2}[\alpha c - \gamma(a-1)](z_{n-1} + 1) + \frac{1}{r-2}[\gamma(d-1) - \alpha c] z_n, \tag{72}$$

where $\alpha,\ \beta,\ \gamma$ and δ are given by:

$$\begin{aligned} \alpha &= \frac{1}{(r-2)(r+1)}[-e(d+1) - frd + gb + hbr], \\ \beta &= \frac{1}{(r-2)(r+1)}\left[edr + f\big(d(r^2-1) - 1\big) - gbr - hb(r^2-1)\right], \\ \gamma &= \frac{1}{(r-2)(r+1)}[ec + fcr - g(a+1) - har], \\ \delta &= \frac{1}{(r-2)(r+1)}\left[-ecr - fc(r^2-1) + gar + h\big(a(r^2-1) - 1\big)\right]. \end{aligned} \tag{73}$$

This step completes the solution of quantum map (51).

6. Concluding remarks

1. The improvement of the estimations in Section 3, Fig. 4 demands exponential improvement of the computer capabilities as is the usual case of chaos [8].
2. The quantization of the torus automorphism has been discussed in [4], but the solutions of the Heisenberg equations were not obtained.

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